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A Study of Diagnostic and Remedial Techniques Used by Master Algebra Teachers

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19 ABSTRACT (Continue on reverse if necessary and identify by block number) This research note raises the issues of what makes for effective diagnosis and remediation of linear algebra equations, and how this affects the development of intelligent tutoring systems. The note reports three studies. In the first, four experienced teachers were given a series of incorrectly worked algebra tasks and asked to provide diagnosis and remediation (n.b. the students were not present). The second study was a series of interviews with three Irish math teachers discussing their approaches to algebra diagnosis and remediation. The third study observed a teacher remediating eight students on the basis of diagnoses provided by the PIXIE (ITS) program. We noticed that this teacher probed for causes beneath the surface errors made by the student. The major conclusions of the three studies were that teachers generally taught algebra procedurally rather than conceptually, and that teachers thought it important to determine the causes behind errors.					
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1. OVERVIEW OF STUDIES

In this paper we raise the issues: what makes for effective* diagnosis and remediation of linear algebraic equations, and how do these issues relate to the development of intelligent tutoring systems? As a basis for this discussion we report two structured interviews and a case study. After introducing each study, we discuss each under the headings of diagnosis and remediation.

Introduction to Study A: the first structured interview. Three male teachers of algebra and one female mathematics teacher's aide, all from school districts in the San Francisco area, served as subjects. They had each taught algebra for between 10 and 40 years.

We presented each, individually, with a copy of a questionnaire (see Appendix A) of algebra tasks and a student's incorrect solutions to the several tasks, and asked them to diagnose the student's error(s) for each item and to suggest remediation. They were free both to spend as much time as they wished on any item and to look over any set of items to check for patterns in the students' solutions. (Please note that the teachers did not have students present).

We compared their diagnoses to the known incorrect procedures used by the students (determined from interview work, Sleeman, 1986), and analysed their suggestions for remediation.

Introduction to Study B: The second structured interview: In order to see if the approaches and opinions of the teachers in Studies A and C of this paper, and of Sleeman, Kelly and Grant (1985), generalised to a

*We studied the behaviour of experienced teachers diagnosing and remediating and assumed that because of their experience, they were effective. No attempt was made to measure their effectiveness objectively.

different culture and school system, the first author interviewed three Irish secondary school mathematics teachers. Two of the teachers had over 20 years of experience and the third had over 4; each teacher had experience teaching algebra to all 7 years in the school. The interview was based on the questionnaire used in Study A (see Appendix A) and around the issues that arose from that study.

Introduction to Study C: a case study in remediation. In Study C, an algebra teacher from a high school in San Francisco area was observed while remediating the algebra errors of eight students. The algebra errors in this study were diagnosed by the PIXIE program (Sleeman, 1982) and were available to the teacher as a basis for his remediation. The teacher was shown how to interpret the printout. The goal of the study was to abstract a model of remediation from this teacher's interaction with the several (8) students. While the focus of study C was remediation, we learned some important facets about diagnosis from it. A limitation of this study is that a post-test was not possible due to time-tabling difficulties at the school.

We shall now discuss the three studies, first under the heading of diagnosis and then remediation.

2. DIAGNOSIS

Diagnosing the "what" of algebra errors - A means-ends search.

An analysis of the protocols of Study A suggests two major findings concerning the teachers' diagnostic strategy: a) that the teachers used a, (GPS) General Problem Solver-type, means-ends search (Newell & Simon, 1963), and b) that searching for patterns across items is a powerful heuristic for diagnosing known student errors.

A GPS-like algorithm reasons forward from the initial state of the problem towards the goal state reducing the "difference" between them by testing appropriate intermediary steps (Newell & Simon, 1963). The evidence for a GPS-type search in the present study was strongly suggested by the protocols of the majority of the teachers. Figure 1 gives an example of such a search tree:

Insert Figure 1 about here

The teacher whose protocol is summarized in figure 1 made three attempts to reduce the "difference" between the equation and the student's solution by choosing substeps (nodes M,N,O; nodes M,P and nodes M,Q,R,S). For example, he seemed to believe that $43/7$ (node S) was not "close enough" to 11 to make that path plausible. He finally opted for the route M-T-U-V-Z. (Note that the teacher appears to have taken a large step to "move" from nodes V to Z).

In Study B, the Irish teachers showed evidence for a similar means-ends

search in diagnosis. A typical statement noted with this task was: "How did he get from there (the equation) to there (the solution)? Let's see, he might have subtracted the 5... No that won't work. Maybe he divided by the 3, and then added 2?" etc.

Not all teachers were practiced at this method of diagnosis, thus some items were left partially diagnosed or undiagnosed. In fact, both groups considered the task of diagnosis, as presented, "artificial" and stressed that they would insist on seeing all of the student's workings of a task. However, even under those conditions students will sometimes turn in incomplete workings of tasks, in which case some form of means-ends analysis would be necessary for a complete diagnosis.

In these studies we presented teachers with a series of equation-student-answer pairs and asked them to make diagnoses. It might reasonably be argued that these teachers had no options but to perform a means-ends search. However, below we list several other possibilities which could be used by a diagnostician, namely:

1. Given the initial form of the equation (e.g., one containing brackets) and the form of the correct answer (say an improper fraction) teachers might create a set of anticipated answers.
2. Recall the set of most frequently made errors in a domain and check if any of these would explain the observed student errors.
3. Given the task, can the student's answer be achieved by manipulating the coefficients in the equation by all known operators (a variant on the means-ends guidance approach).
4. Create the correct solution path, and subsequently include all known incorrect versions of the rules and incorrect orderings (this is

what PIXIE does), and check if any of these explain the student's answer.

and the default:

5. If no diagnosis could be found (by using any of the methods including means-ends guidance) then conclude that the student needs to be retaught (parts of) the skill.

Searching for patterns of errors. In Study A, it was generally the case that the teachers (one in particular) who searched for patterns across items were the ones who were more successful at diagnosis, but, of course, searching for patterns did not guarantee finding them.

In our analysis, we labelled an attempt to diagnosis in which the teacher found at least one complete path to the solution (i.e., gave a full set of steps to explain a student's error as a complete search; we labelled other attempts incomplete searches). For example, the items in Set 2 (See Appendix A) were of the general form $ax = bx + c$ with the student's incorrect solution $x = (a + b + c)/2$. Several teachers who gave incomplete searches stopped when they reached the simplification of the solution in the general form $x = a + b + c$, unable to explain the student's division by 2. (In this case, the student had incorrectly added all the coefficients, and divided by 2 because there were two X-terms in the original equation (Sleeman 1986)).

We further sub-classified "complete" searches as matching searches (those that matched the known error patterns of the students (from Sleeman, 1986)) and alternative searches (those that were plausible alternative explanations for the error(s)). From these a metric for a teacher's skill at finding the students' known error patterns was developed (see Table 1).

Insert Table 1 about here

Overall, the teachers gave complete searches for an average of 16 of the 23 items, gave incomplete searches for an average of 3 items, and offered no diagnosis whatsoever for an average of 3 items. (Items with no diagnosis were classed by the teachers as "unusual". For example, the items in Set 4 were difficult for three of the teachers because the student's solutions contain two different values for X , even though the equation in each case is linear).

For example, Teacher 4 had the highest percentage of both complete and matching searches.

The average number of matching searches was just under 5 for the remaining three teachers. For Teacher 4, it was thirteen - due in part to his skill in finding common error patterns in a set. He would, typically, diagnose the error of the first item and check if this diagnosis also explained the errors in the remaining items in the set. If not, he would iterate this process until he found a common error pattern, or determined to his satisfaction that there was none. He searched persistently for a pattern; moreover, he did not allow himself to be daunted by the absence of a pattern underlying the items in Set 3. Consequently, he found the patterns in Sets 4 and 5 -- underscoring the importance of continuing to search for patterns of errors in the face of disconfirming evidence.

On the other hand, teachers who typically looked at diagnosis on a per item basis usually gave incomplete searches, or terminated the search

when a feasible solution path was found (i.e., an "alternative search"). Such behaviour may lead to the reporting of some superficial solution paths. For example, a student's solution of $X = 6$ for the equation $3 + 4X = 18$ was explained as the student's "forgetting" the "+4" and solving the remaining equation, $3X = 18$. (In this case the student's error was actually the result of misinterpreting the plus sign as a multiply sign (which gave $12X = 18$), and then subtracting 12 from both sides.)

Diagnosing the "why" of algebra errors - taking diagnosis a step deeper

The teacher in Study C (i.e., the one who tutored eight students based on PIXIE's diagnoses) believed that many different causes underlie what appears, superficially, to be the same error. He searched for a causal underpinning (a why) for the syntactical error (the what). For him, diagnosis was not complete without a causal explanation. This can be illustrated by how the teacher handled three students who displayed the same mal-rule namely, inverting the final fractional solution. This teacher concluded it was unfamiliarity with improper fractions for two students; for the third it was a misunderstanding of the mathematical notation for expressing fractions.

His method of diagnosis was to present the student with a simplified version of the equation, observe the method used, and from this information infer a reason for the error noted. For example, he inferred errors due to: a) an algebraic procedure with limited application; b) misunderstanding fractional notation-A, c) misunderstanding fractional notation-B; and d) unfamiliarity with improper fractions.

a) Errors due to an algebraic procedure with limited application:

In this class of errors, the teacher concluded that the error identified by PIXIE was caused by the student using an algebraic procedure that had

only limited application. It worked for simple equations, but caused errors in more difficult ones.

For example, in the case of student A3, PIXIE indicated the student was having difficulties with equations of the form $3X = 5$. To ascertain the reason behind the error, the teacher set the student the simpler equation $3X = 6$ (simpler because it gives an integer solution). The student solved it by saying "3 times 2 is 6." From this, the teacher concluded that the student could not solve an equation like $3X = 5$, since the student couldn't think of what number times 3 gives 5. The procedure of looking for a whole number to substitute for X in order to balance both sides of the equation had only limited application and foundered when the value for X was a non-integer. The teacher verbalised his reasoning about the error:

(For the equation $6X = 12$, he said:) "Some people ask: '6 times what number is 12?'. When you are used to doing it this way, it is hard to do it when you are dealing with a fraction. You need a method that will always work."

b) Errors due to misunderstanding fractional notation-A:

One student appeared to have a general procedure for solving an equation, but because of unfamiliarity with fractions had difficulty expressing the solution. For example, given the equation $9X = 6$, his solution was $X = 1$. He could perform the rule "divide each side by 3" and obtain $3X = 2$. However, confused by the format for fractional solutions ($2/3$), he instead subtracted the 2 from the 3. The teacher pointed out that the student's problem was not primarily algebraic, but mistaking the notation of fractions (for that of subtraction). (And supposedly following the subtraction bug of "subtract the smaller number

from the bigger" (Brown and Burton, 1978)).

c) Errors due to misunderstanding fractional notation-B.

In this error, the teacher believed the student knew how to solve an equation of the form $aX = b$ by dividing both sides of the equation by a , but wrote the final solution as a/b , due to a misunderstanding of the mathematical notation of the fraction.

For example, the teacher set student A2 the equation $3X = -2$. He wrote $X = -3/2$. The teacher asked the student how he would represent "5 divided by 4." He wrote "4/5."

d) Errors due to an unfamiliarity with improper fractions

In this class of errors, the teacher believed the student to have a general procedure for obtaining the solution, but, being unfamiliar or uncomfortable with improper fractions, the student expressed the solution as a proper fraction instead. For example (student A1), the equation was $9X = 16$ to which the student gave the solution $X = 9/16$. The teacher responded "You are used to getting fractions less than 1. You might want to write $X = 9/16$, just because it looks better (than $16/9$)". This type of error might also indicate the student's tendency to regress to earlier methods when faced with new problems (Davis, Jockusch & McKnight, 1978).

Summary: Diagnosis

To summarise this section on diagnosis, we can see that under conditions of limited information, teachers are likely to use a means-ends search to discover the incorrect solution path of the student, and that a tendency to look for error patterns, if consistently applied, often leads to good diagnosis. We have also seen that diagnosis of error paths may

be just half the story - determining the reason for the error may also be important.

3. DECIDING ON APPROPRIATE REMEDIATION

The need for a detailed diagnosis to serve as a basis for remediation is a basic assumption underlying much of the work in intelligent tutoring systems - e.g. DEBUGGY (Brown & Burton, 1978) and LMS/PIXIE (Sleeman, 1982). However, many teachers' approaches to remediation may not be guided by this perspective.

The teachers in Study A sometimes suggested remediation for fewer errors than they diagnosed (see Table 2).

Insert Table 2 about here

The teachers appeared to review their diagnoses and make some summary judgement about the student's difficulties and then begin remediation. They rarely remediated more than one issue per task, and their suggestions ranged from vague statements that advised the student to go "Back to basics," through procedural prescriptions like "Get rid of added things first," to conceptual calls to "Show the student the difference between Xs and numbers." Generally speaking, the teachers favoured rule-based over "conceptual" remediation (see Table 3).

Insert Table 3 about here

Only one of the four teachers consistently referenced the current algebra item in his remediation - the others gave more general feedback (see Table 4).

Insert Table 4 about here

Some diagnosed errors may have been ignored because these did not fall under the scope of an agenda triggered by the task (Putnam, 1987). According to Putnam (1987), a sample of second-grade teachers working with both actual and simulated students did not probe for a detailed diagnosis before they began to reteach the topic. Thus, reteaching was often at a general level, and did not always reference the task on which the student had encountered difficulty. Putnam suggests that such teachers are following some script-based agenda and that a perceived weakness in an area "triggers" part of that agenda. Alternatively, these errors may not have been judged as critical for understanding algebra.

The teachers in Study B said that they would remediate fewer errors than they had diagnosed for motivational reasons, particularly in cases of low-ability students. They believed that pointing out successes in mathematics was more effective in the long run to pointing out failures (for related reading, see Fennema and Behr, 1980; Kulm, 1980; Reyes,

1980).

The teachers in Study B further believed that time spent on diagnosing errors due to violation of rules might be better spent remediating what they considered to be more fundamental problems in algebra: considering the X variable to be a "letter" (indicating major conceptual misunderstandings in mathematics — see Davis, 1984; Kuchemann, 1978), misunderstanding of fractions and negative numbers, misapplying earlier knowledge (Davis, Jockusch & McKnight, 1978), and not knowing basic mathematical facts.

What we noted, therefore, among these teachers was a tendency not to develop a detailed diagnosis before beginning remediation.

Study C: The case study

The teacher in Study C seems to be an exception to the above trend in that this teacher's remediation was based not only on the diagnosis of the syntax of the error, but also on the reason behind the error. His approach to remediation can be outlined as follows:

After diagnosing the what and the why of the error, the teacher:

- a) Referred indirectly to the error.
- b) Reaffirmed the correct procedure. (Note: the teacher did not explicitly indicate when to apply this procedure).
- c) Reassured the student that the new method gave acceptable solutions.
- d) Gave additional instruction.
- e) Gave practice items.

We shall now discuss each of these steps in some more detail.

a) Referring indirectly to the error:

The teacher pointed out the students' errors in an indirect fashion (as if not lessen the students' motivation for mathematics c.f., the teachers in Study B). The teacher in Study C used techniques such as a) universalising the error by claiming that it was common, (e.g., to student A2, who worked $3X = 2$ as $X = 3/2$, he said "You make a common mistake that many people do"). (This tendency to universalise the error is interesting, since the universalisation of errors is known to be one of the curative factors of group psychotherapy (Yalom, 1980)). b) Other techniques included remaining tentative in assigning blame to the student for the error (e.g. for the student who wrote $X = 9/16$ for $9X = 16$, the teacher said, "You might want to write $X = 9/16$, just because it looks better"), or c) incriminating some undefined "others" for it (e.g., for the student who wrote $X = 3/2$ for $3X = 2$, the teacher said, "Would you believe that some people would write $3/2$?"), and d) allowing the student an excuse for the behaviour (e.g., to one student he said, "You had some problems with this ($6X = 9$). It seems that you subtracted 6 from 9. You might have been confused. It doesn't work. It might look like a good answer").

b) Reaffirming the correct procedure. Once the teacher had pointed out an error to the student, he set about reaffirming the correct procedure. For example, to student A4 he said, "The procedure of dividing across by 3 in $3X = 6$ should be the same for $3X = 5$, no matter if you get nice numbers or not."

To student A1, concerning an equation of the form $aX = b$, he said, "Even if this number b is bigger than this number a, the procedure (dividing both sides by a) is still the same."

c) Reassuring the student that the new method gives acceptable solutions. The teacher typically reassured the students that the 'unusual' solutions produced by the now-reaffirmed correct procedures were indeed acceptable.

To the student (A2) uncomfortable with fractional solutions, he said, "Don't let it (the fraction) bother you. Two-thirds is a good number. There are a lot of fractions in the world." To those uncomfortable with improper fractions (eg, student A4), he said, "9/6 is a number (the solution to $6X = 9$). Is it a whole number? No. But it is still a number. This is a legal number. These numbers exist in the world. We don't give up because the answer is a fraction. Let's figure out the best way to write this number (reduces it to $3/2$). This is not a nice number. The numerator on top is bigger than the denominator. Fractions come in all shapes."

d) Giving additional instruction. If it were called for, the teacher presented new material during remediation. For example, he explicitly labelled fractions "proper" or "improper". He showed some students how a fraction a/b (expressed here in a general form) was simply another way of writing $a \div b$; showed another student that division was really a case of repeated subtraction. And, to a final student, two procedures for solving the same equation (dividing by the X coefficient or multiplying by its inverse).

e) Giving practice items. The teacher in Study C gave only three of the eight students practice items (one to each student). The item required the student to demonstrate a grasp of the reaffirmed procedure. The number of practice items assigned in this study was small, but each of the three teachers in Study B stressed the importance of giving many practice items to each student. In fact, they believed that one error

was enough for the student to handle at a time and would assign a full set of examples to drive one point home (a similar approach was recommended by Buckingham, 1933). A series of studies need to be undertaken which attempts to determine the importance of these several steps for Remediation.

Rule-based vs. "conceptual" instruction or reinstruction

Perhaps the approach to diagnosis and remediation would differ markedly if algebra was taught conceptually (rather than procedurally as in these studies). A classic division of instructional approaches may be labelled the "conceptual vs rule-based" division, which has fueled debate on instruction in mathematics since at least the turn of the century (e.g., Beatley, 1954; Byers, 1980; Cronbach & Snow, 1977; Davis, 1984; Eisenberg, 1975; Godfrey, 1910; Ormell, 1976; Skemp, 1976).

The emphasis on using rules to tutor in algebra was noted, with some exceptions, in the structured interviews, in the case study, and in Sleeman, Kelly and Grant (1985). In Study B, the three teachers indicated that the students who required extensive remediation in algebra were ones for whom a conceptual remediation was inappropriate. They felt that such low-ability students were better served, in the time allowed within an exam-oriented system, by being given a small number of hard-and-fast rules.

These teachers noted additional advantages to a rule-based approach including relatively straightforward structuring of lesson units and individual lesson plans. Using a procedural approach, one had a finite set of rules to teach together with a mechanical directive to "do the same thing to both sides" which was seen as "easier to teach" than the many metaphors, illustrations, etc. required in a conceptual approach,

(see also Sleeman et al., 1985). Finally, the teachers in Study B believed that "success" in teaching was easily measured when based on the learning of rules. One could discover what rules had and had not been learned. Remediation and further instruction within this framework were then "clear".

Difficulties stemming from the wide range of individual differences among their students was a concern for both the American and Irish teachers (on the topic of individual differences and mathematics, see Carry, 1983; Cronbach & Snow, 1977; Pennema & Behr, 1980; Snow, 1983; Threadgill-Sowder, 1985). For example, an Irish teacher cited Piagetian research to argue that many students were ill-prepared for conceptual-based instruction in mathematics (on this general point, see Adi, 1978; Grady, 1976; Lovell, 1972; Mehlhorn, 1981).

We do not wish to take sides on this debate, rather we simply wish to point out that among the teachers we have seen, the practice of teaching algebra as a set of rules is widespread, and is often justified on grounds of favouring "weak" students and on the grounds of "efficiency"; for dissent on these final points, see (Brown, 1982).

4. SUMMARY OF EXPERIMENTS AND IMPLICATIONS

The three studies reported above were undertaken to help determine how PIXIE - an ITS that has the capability of diagnosing errors and an embryonic capability of remediation - might be modified to perform these activities more effectively. The following section summarises the conclusions from the experiments and discusses their implications for PIXIE.

In summary, we may draw three tentative conclusions from these experiments: a) In the absence of a student, teachers in Study A generally

used a means-ends approach to diagnosing errors; amongst this group, the most successful diagnostician searched for patterns; b) The majority of the teachers used a procedural rather than a conceptual approach to remediating algebra; c) The teacher in Study C did not take a mal-rule at its face value, but probed the student to verify the mal-rule and to determine its cause, and based remediation on this combined information.

Implication for Pixie: We shall briefly address the implications for PIXIE arising out of each of the above points: a) PIXIE is able to do a much more thorough analysis of the search space than (these) teachers. PIXIE's model-generation creates models with all known mal-rule variations and all order-sensitive pairs of rules. Thus, PIXIE may well be expected to produce a better diagnosis than a teacher, who might well be limited in the number of potential solution paths that could be held in memory at any one time.

b) It is basically good news for PIXIE that teachers take a predominantly procedural approach to instruction and remediation as it is relatively straightforward to produce a procedural explanation for a student. (It is analogous to giving a trace of the steps undertaken by PIXIE).

c) Probing the nature of a student's misunderstanding is likely to require the student to give an explanation for an action in natural language. Natural language interfaces seem only to be effective if one has a well-constrained domain in which the vocabulary is severely limited. However, a natural language interface to allow a student to describe the several steps in a non-deterministic algorithm has been implemented (Sleeman & Hendley, 1982).

The above conclusions and implications are drawn tentatively, given the

number of teachers involved in the studies. In addition, work recently completed by this research group questions whether a detailed diagnosis, with or without known causes for the mal-rules, is a necessary prerequisite for effective remediation.

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Appendix A

The first 2 pages of the questionnaire are identical to those given to the teachers. To cut down on space, we have then merely listed the remaining task-student-answer pairs. In the actual questionnaire each task-student-answer pair appeared on a separate page; each page having the same format as the one containing task-student-answer pair 1.1.

School of Education

Stanford University

Selected Algebra Problems with Student Answers

The following are examples of students' difficulties in working algebra problems. Please work through each problem to see if you can identify the error the student is making in each case. (Note: this is a genuine exercise as we currently have no explanation for some/many of the errors recorded).

Thank you for your help.

D Sleeman/Eamonn Kelly

6 December 1984

Algebra Problems

Set 1

1.1 $3+4x=18$

$x=6$

A. What do you think the student is doing wrong to get this answer?

B. How would you remediate this problem?

1.2	$6+4x=32$	$x=8$
1.3	$5+3x=27$	$x=12$
1.4	$9+3x=40$	$x=13$
1.5	$4+7x=39$	$x=11$
2.1	$3x=2x+7$	$x=6$
2.2	$6x=3x+5$	$x=7$
2.3	$5x+4x=11$	$x=10$
2.4	$3x=8x+6$	$x=17/2$
2.5	$4x=2x+8$	$x=7$
3.1	$2+5x=15$	$x=1/25$
3.2	$3+4x=18$	$x=18/7$
3.3	$4+6x=20$	$x=1/22$
3.4	$5+3x=20$	$x=12$
3.5	$3+2x=8$	$x=48$
4.1	$2x+3x=10$	$x=1, x=3$
4.2	$3x+2x=11$	$x=2, x=3$
4.3	$3x+2x=13$	$x=2, x=5$
4.4	$2x+4x=14$	$x=4, x=2$
4.5	$2x+3x=10$	$x=3, x=1$
5.1	$7x+5x=16$	$x=2$
5.2	$3x+4x=20$	$x=\sqrt{13}$
5.3	$10x+4x=21$	$x=\sqrt{7}$

Table 1

Overall analysis of the search techniques used by the 4 Teachers

	% of items searched for patterns of errors	# of "complete"* searches (% in parentheses)	# of "matched"* searches (% in parentheses)
Teacher 1	26	15(65)	6(26)
Teacher 2	0	15(65)	3(13)
Teacher 3	47	17(74)	5(22)
Teacher 4	81	19(83)	13(57)

*Percentages (in parentheses) based on 23 items.

Table 2

Items for which the number of errors diagnosed
did not match the number remediated.

	Items for which a "complete" search path was found (out of 23)	Items in which all errors were remediated	Items in which partial remediation was given
Teacher 1	15	9 (60%)	6 (40%)
Teacher 2	15	13 (87%)	2 (13%)
Teacher 3	17	13 (76%)	4 (24%)
Teacher 4	19	11 (58%)	8 (42%)

Note: Table 2 shows the number of items which were completely and partially remediated. Deciding upon the actual numbers of errors within items is a subjective judgment, and hence the numbers of errors cannot meaningfully be quoted.

Table 3

A categorisation of the teachers' suggestions* for remediation

(As percentage of total suggestions per teacher)

	Concept-based	Rule-based	Other
Teacher 1	20	53	27
Teacher 2	50	20	30
Teacher 3	0	98	2
Teacher 4	28	56	16

*Note: Concept-based: suggestions of the sort "Discuss the idea of a variable". Rule-based: suggestions of the sort "Take the Xs to the left-hand side". "Other: classification" vague statements such as "Start over".

Table 4

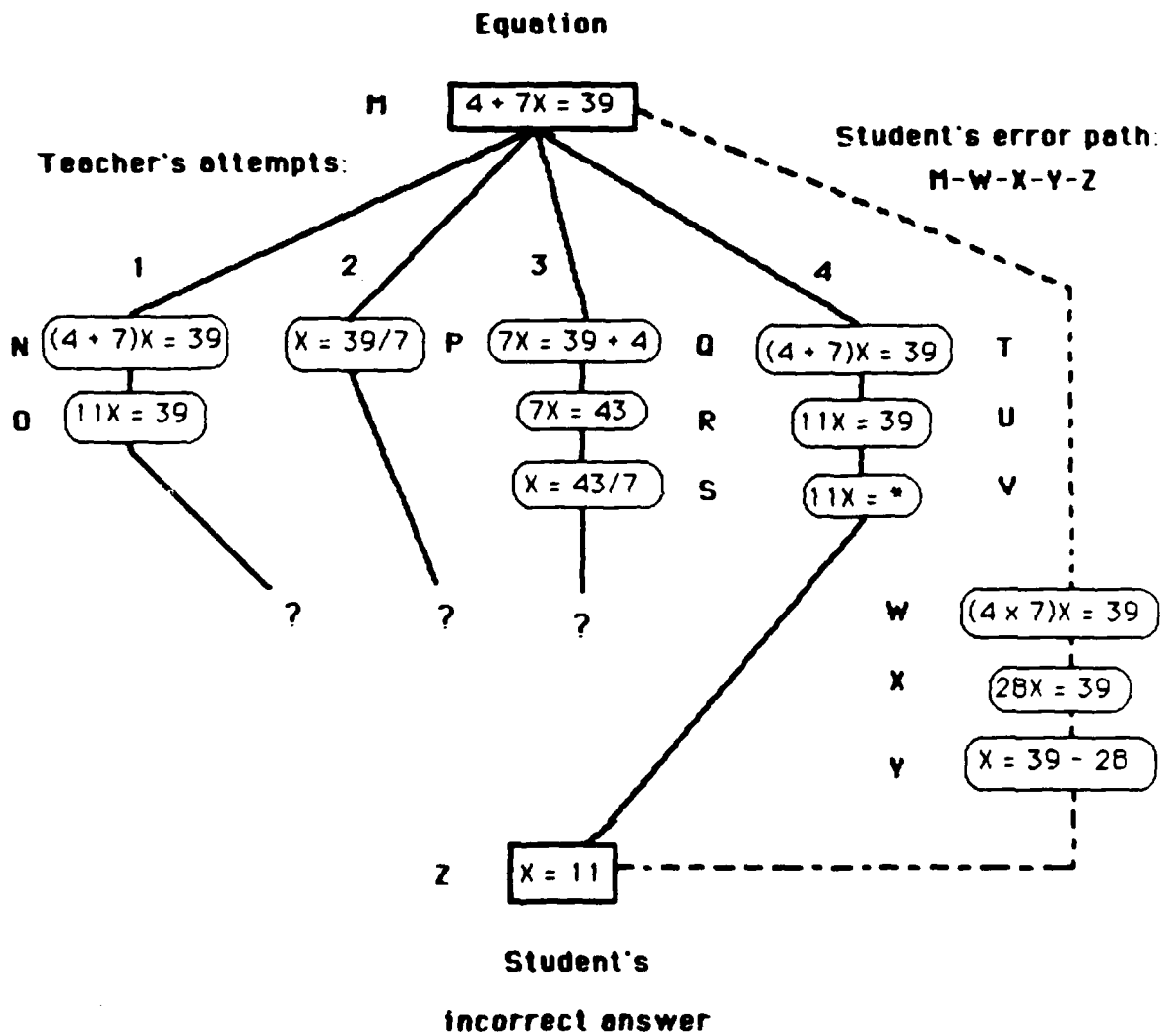
Referencing the student's work during remediation.

Percentage of remedial suggestions that referenced the:

	Equation	Substeps/solution	No reference
Teacher 1	73	0	27
Teacher 2	10	0	90
Teacher 3	18	60	22
Teacher 4	18	0	82

Figure 1

Example of a GPS-type Search
of a Solution Tree



* The teacher said, "The student ignored the 39 and wrote $X = 11$."